THE SIMPSON'S RULE FOR FRACTIONAL INTEGRAL OPERATORS

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1 Introduction

In this paper, we propose an approach based on quadratic interpolation to the numerical evaluation of the composition of the left and right Riemann-Liouville integrals. The presented methodology is a fractional equivalent to the classical Simpson's rule [1].

2 Fractional preliminaries

In this section, we introduce the fractional operators used in this work. According to the fractional calculus [3, 4] we recall the definitions of the left and right Riemann-Liouville fractional integrals for $\alpha > 0$

$$I_{a^+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \quad (t > a) \tag{1}$$

$$I_{b^-}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}} d\tau \quad (t < b) \tag{2}$$

where denotes the Gamma function. Fractional integral operators, which are a composition of the left and right fractional Riemann-Liouville integrals, look as follows (see [2])

$$\mathcal{I}_{a^+,b^-}^{\alpha,1} f(t) := I_{a^+}^{\alpha} I_{b^-}^{\alpha} f(t), \quad \text{for } t \in [a,b] \tag{3}$$

$$\mathcal{I}_{b^-,a^+}^{\alpha,1} f(t) := I_{b^-}^{\alpha} I_{a^+}^{\alpha} f(t), \quad \text{for } t \in [a,b] \tag{4}$$

3 Main results

The interval $[a, b]$ is divided into N (even) sub-intervals $[t_i, t_{i+1}]$, for $i = 0, 1, ., N-1$ with a constant step $\Delta t = (b-a)/N$ by using nodes $t_i = a + i\Delta t$. Next, we replace function f by the quadratic polynomial, which takes the same values as at the end points t_{2j} and t_{2j+2}, and the midpoint t_{2j+1}

$$\begin{aligned} f(\tau) \approx & \frac{(\tau-t_{2j+1})(\tau-t_{2j+2})}{2(\Delta t)^2} f(t_{2j}) - \frac{(\tau-t_{2j})(\tau-t_{2j+2})}{(\Delta t)^2} f(t_{2j+1}) \\ & + \frac{(\tau-t_{2j})(\tau-t_{2j+1})}{2(\Delta t)^2} f(t_{2j+2}) \end{aligned} \tag{5}$$

We put the interpolation (5) into expressions (1)-(2) and by the additivity of integration we get the approximations of analysed fractional operators.

$$\begin{aligned} & I_{a^+}^{\alpha} f(t)\big|_{t=t_i} \\ & \approx \frac{(\Delta t)^{\alpha}}{2\Gamma(\alpha)} \sum_{j=0}^{\frac{i-2}{2}} \Big\{ \left[f_{2j} - 2f_{2j+1} + f_{2j+2}\right] \left[i^2 v_{i,j}^{\alpha} - 2i v_{i,j}^{\alpha+1} + v_{i,j}^{\alpha+2}\right] \\ & - \left[(4j+3) f_{2j} - (8j+4) f_{2j+1} + (4j+1) f_{2j+2}\right] \left[i v_{i,j}^{\alpha} - v_{i,j}^{\alpha+1}\right] \\ & - \left[\left(4j^2+6j+2\right) f_{2j} - 8j(j+1) f_{2j+1} + 2j(2j+1) f_{2j+2}\right] i v_{i,j}^{\alpha} \Big\} \\ & = S^L(t_i, \Delta t, f_i, \alpha) \end{aligned} \tag{6}$$

where

$$v_{i,j}^{\beta} = \begin{cases} 0 & \text{for } i = j = 0 \\ \dfrac{(i-2j)^{\beta} - (i-2j-2)^{\beta}}{\beta} & \text{otherwise} \end{cases} \tag{7}$$

and

$$\begin{aligned} & I_{b^-}^{\alpha} f(t)\big|_{t=t_i} \\ & \approx \frac{(\Delta t)^{\alpha}}{2\Gamma(\alpha)} \sum_{j=\frac{i}{2}}^{\frac{N-2}{2}} \Big\{ \left[f_{2j} - 2f_{2j+1} + f_{2j+2}\right] \left[i^2 u_{i,j}^{\alpha} + 2i u_{i,j}^{\alpha+1} + u_{i,j}^{\alpha+2}\right] \\ & - \left[(4j+3) f_{2j} - (8j+4) f_{2j+1} + (4j+1) f_{2j+2}\right] \left[i u_{i,j}^{\alpha} + u_{i,j}^{\alpha+1}\right] \\ & - \left[\left(4j^2+6j+2\right) f_{2j} - 8j(j+1) f_{2j+1} + 2j(2j+1) f_{2j+2}\right] i u_{i,j}^{\alpha} \Big\} \\ & = S^R(t_i, \Delta t, f_i, \alpha) \end{aligned} \tag{8}$$

where

$$u_{i,j}^{\beta} = \begin{cases} 0 & \text{for } i = j = N \\ \dfrac{(2j-i+2)^{\beta} - (2j-i)^{\beta}}{\beta} & \text{otherwise} \end{cases} \tag{9}$$

The numerical results obtained for operator $\mathcal{I}_{1^-,0^+}^{\alpha,1} f(t)$ and order $\alpha \in \{0.4, 0.6, 0.8, 1, 1.5, 2\}$ are presented below

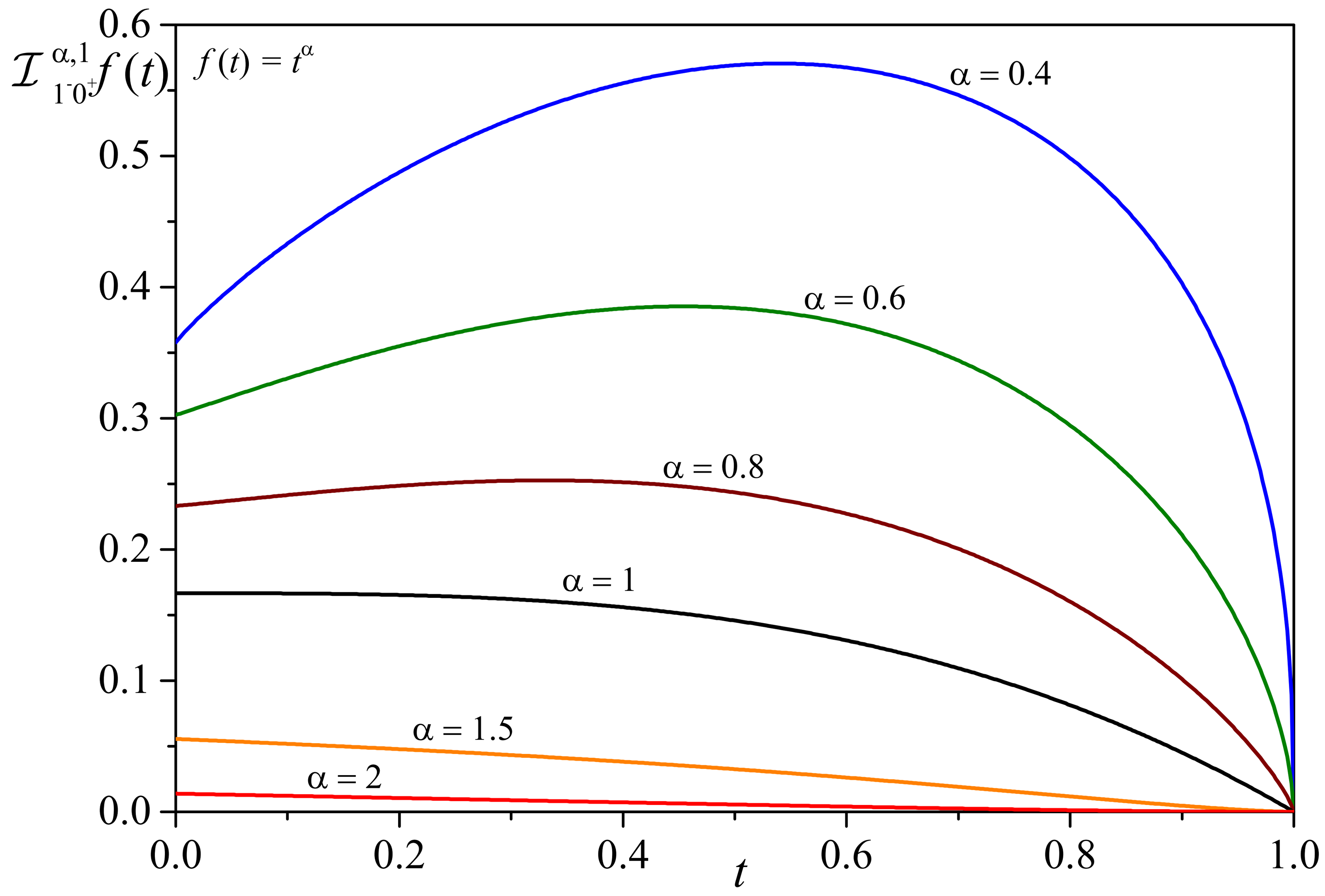

Figure 1: Numerical evaluation of the integral operator $\mathcal{I}_{1^-,0^+}^{\alpha,1} f(t)$ for different values of α.

Table 1: Maximum errors generated by the described method for the integral $\mathcal{I}_{1^-,0^+}^{\alpha,1} t^{3-\alpha}$

Δt	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.99$
1/10	$4.89 \cdot 10^{-5}$	$3.33 \cdot 10^{-5}$	$1.65 \cdot 10^{-5}$	$5.99 \cdot 10^{-6}$	$2.31 \cdot 10^{-7}$
1/20	$5.34 \cdot 10^{-6}$	$3.25 \cdot 10^{-6}$	$1.45 \cdot 10^{-6}$	$4.93 \cdot 10^{-7}$	$1.95 \cdot 10^{-8}$
1/40	$5.78 \cdot 10^{-7}$	$3.18 \cdot 10^{-7}$	$1.26 \cdot 10^{-7}$	$4.05 \cdot 10^{-8}$	$1.72 \cdot 10^{-9}$
1/80	$6.32 \cdot 10^{-8}$	$3.09 \cdot 10^{-8}$	$1.08 \cdot 10^{-8}$	$3.36 \cdot 10^{-9}$	$1.63 \cdot 10^{-10}$
1/160	$6.98 \cdot 10^{-9}$	$2.99 \cdot 10^{-9}$	$9.26 \cdot 10^{-10}$	$2.84 \cdot 10^{-10}$	$1.65 \cdot 10^{-11}$

4 Conclusions

In this paper new formulas for numerical calculation of fractional integrals were presented. We derived the numerical schemes for the left, and the right Riemann-Liouville fractional integrals, and for the composition of these operators, using quadratic interpolation. Finally, examples of numerical evaluations of analyzed operators of selected function and maximum errors generated by the described method are also shown.

References

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